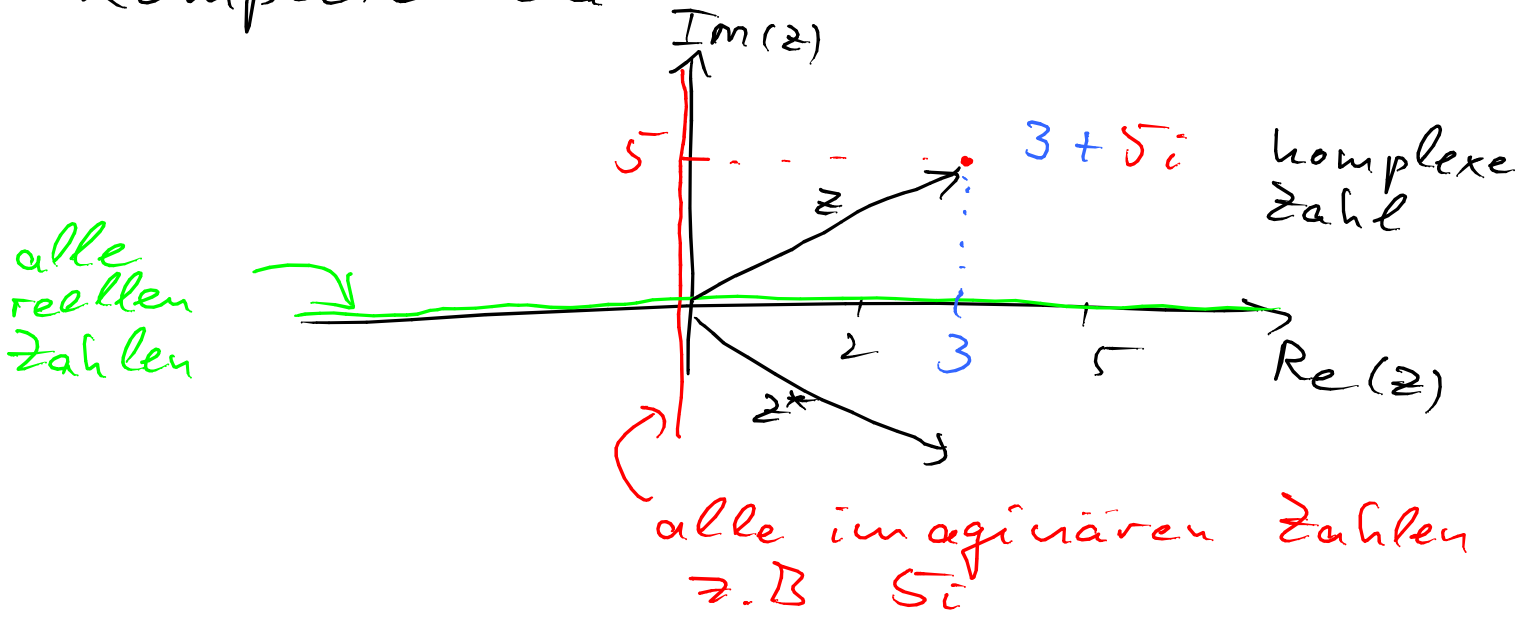
VMA 2 – 18.05.2015

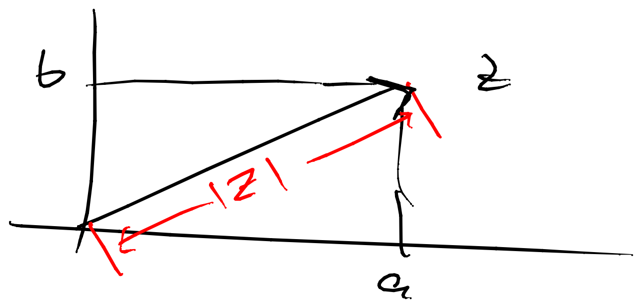
Komplexe Zahlen



z: komplexe Zahl $\quad z = a + ib$

z*: konjugiert-komplex $\quad z^* = a - ib$

Betrag



$$\boxed{|z| = \sqrt{a^2 + b^2}}$$

$$zz^* = (a+ib)(a-ib) = a^2 - (ib)^2 = a^2 + b^2$$

$$\Leftrightarrow \sqrt{zz^*} = \sqrt{a^2 + b^2}$$

Division:

$$z_1 = a_1 + ib_1$$
$$z_2 = a_2 + ib_2$$

$$z = \frac{z_1}{z_2} = \frac{z_1 \cdot z_2^*}{z_2 \cdot z_2^*} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}$$

$$= \frac{a_1 a_2 + b_1 b_2 + i(b_1 a_2 - a_1 b_2)}{a_2^2 + b_2^2}$$

$$\Rightarrow \boxed{z = \frac{a_1a_2+b_1b_2}{a_2^2+b_2^2} + i\,\frac{b_1a_2-a_1b_2}{a_2^2+b_2^2}}$$

Übungen

a) $i(3-2i) = 3i - 2i^2 = \underline{\underline{3i+2}}$

b) $|3-4i| = \sqrt{3^2+(-4)^2} = \sqrt{25} = \underline{\underline{5}}$

c) $\frac{3+4i}{2-i} = \frac{(3+4i)(2+i)}{(2-i)(2+i)} = \frac{6-4+i(4\cdot 2+3\cdot 1)}{2^2+1^2}$

$= \underline{\underline{\frac{2}{5} + i\,\frac{11}{5}}}$

Begründung für Eulersche Formel

$$e^{i\varphi} = \cos\varphi + i\sin(\varphi)$$

über Taylorreihe für $b = i\varphi$

$$e^b = 1 + \frac{b^1}{1!} + \frac{b^2}{2!} + \frac{b^3}{3!} + \dots$$

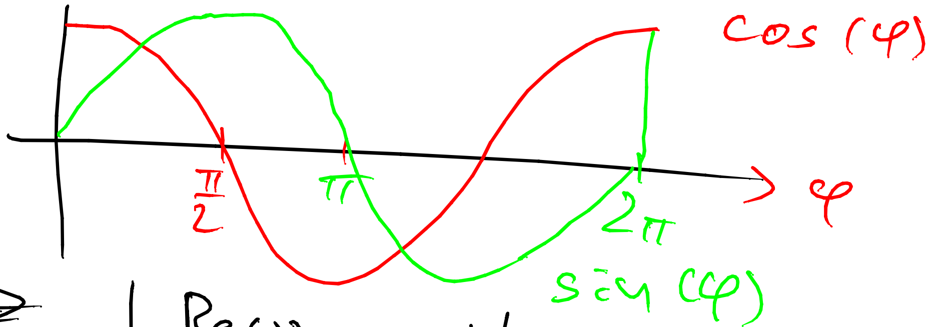
$$e^{i\varphi} = 1 + \frac{(i\varphi)^1}{1!} + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \dots$$

$$= \underline{1} + i\,\underline{\frac{\varphi}{1!}} - \underline{\frac{\varphi^2}{2!}} - i\,\underline{\frac{\varphi^3}{3!}} + \dots$$

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} \dots$$

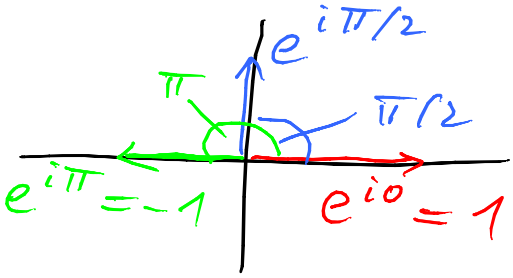
$$i\sin\varphi = i\left(\frac{\varphi}{1!} - \frac{\varphi^3}{3!} + \dots\right)$$

$$\Rightarrow \boxed{\cos\varphi + i\sin\varphi = e^{i\varphi}}$$

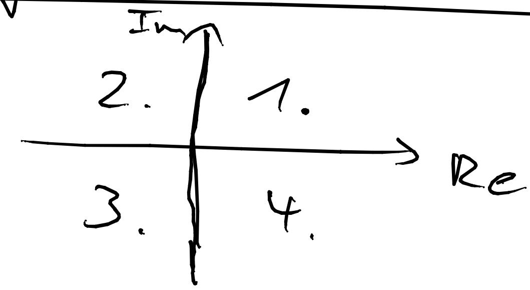


z	$Re(z) = \cos(\varphi)$	$Im(z) = \sin(\varphi)$
e^{i0}	1	0
$e^{i\pi}$	-1	0
$e^{i2\pi}$	1	0
$e^{i\pi/2}$	0	1

$$\boxed{e^{i\pi} = -1}$$



Umrechnung "kartesisch → Polar"

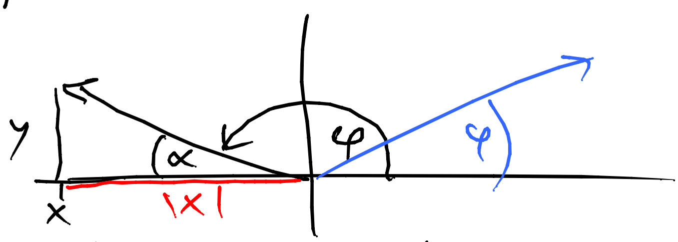


$z = x + iy$

$Re(z) < 0$ | $Re(z) > 0$

$x < 0$ | $x > 0$

2. Quadrant



$$\tan\alpha = \frac{y}{|x|} = \frac{y}{-x} \quad | \operatorname{atan}()$$

$$\alpha = \operatorname{atan}\left(\frac{y}{-x}\right) = -\operatorname{atan}\frac{y}{x}$$

$$\varphi = \pi - \alpha = \pi - \left(-\operatorname{atan}\left(\frac{y}{x}\right)\right)$$

$$\varphi = \pi + \operatorname{atan}\left(\frac{y}{x}\right)$$

<u>Beispiel</u>

$$z = a + ib$$

z.B. $z = 2 + i5$ $\quad \operatorname{Im}(z) = 5$

$z' = 3 - i7$ $\quad \operatorname{Im}(z) = -7$

<u>Beispiel</u>

$z = 3e^{i\,5 \cdot 2\pi/6}$ Polarform

Umrechnung zu kartesisch:

$$x = r\cos\varphi = 3\cos\left(\frac{5}{6} \cdot 2\pi\right) = 3 \cdot \frac{1}{2} = \underline{\underline{1.5}}$$

$$y = r\sin\varphi = 3\sin\left(\frac{5}{6} 2\pi\right) = -\frac{3}{2}\sqrt{3} = \underline{\underline{-2.598}}$$

Umrechnung zu Polarform

Geg. x, y

Ges. r, φ

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\sqrt{3}\right)^2}$$

$$= \sqrt{\frac{9}{4} + \frac{27}{4}} = \sqrt{\frac{36}{4}} = \underline{\underline{3}}$$

Welcher Quadrant? $x > 0,\ y < 0 \Rightarrow$ 4. Quadr.

$$\varphi = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{-\frac{3}{2}\sqrt{3}}{\left(\frac{3}{2}\right)}\right) =$$

$$\Rightarrow \varphi = -\frac{\pi}{3} = -1.047$$

Anders als $\varphi = 5 \cdot \frac{2\pi}{6}$? — Nein

denn $-\frac{\pi}{3} + 2\pi = -\frac{\pi}{3} + \frac{6\pi}{3} = 5\frac{\pi}{3}$

$= 5\frac{2\pi}{6}$