VMA 2 — 09.05.2016

Komplexe Zahlen

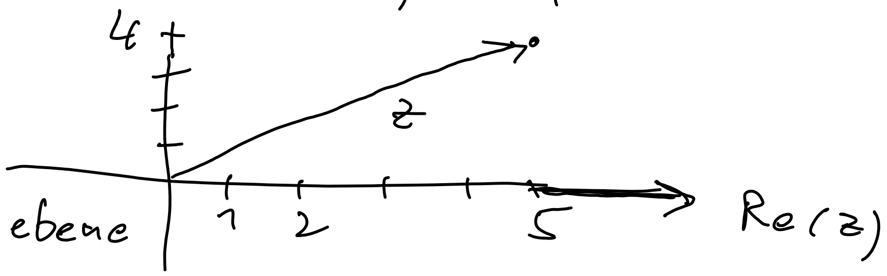
Wdh:

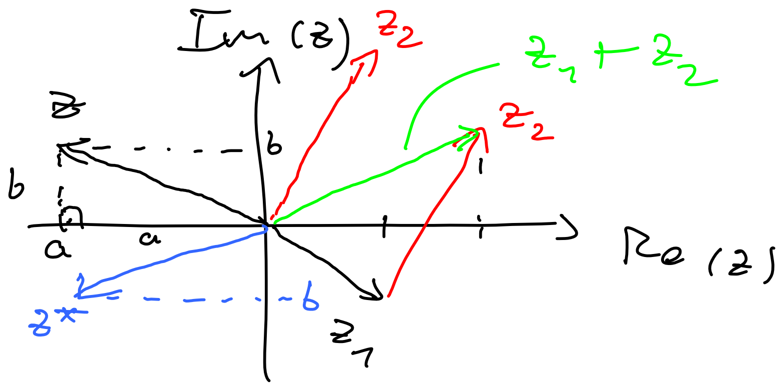
$$z = 5 + 4i$$

Realteil $\mathrm{Re}(z) = 5$

Imaginärteil $\mathrm{Im}(z) = 4$

komplexe Zahlenebene = Gaußsche Zahlenebene





konjugiert komplexe Zahl z^*

$$z = a + ib$$

$$z^* = a - ib$$

Betrag $|z|$ einer komplexen Zahlen (Länge des Vektors z)

$$|z|^2 = a^2 + b^2 \quad \Rightarrow \quad |z| = \sqrt{a^2 + b^2}$$

$$\begin{aligned} z \cdot z^* &= (a+ib)(a-ib) \\ &= a^2 - (ib)^2 \\ &= a^2 - i^2 b^2 \\ &= a^2 + b^2 = |z|^2 \end{aligned}$$

Beispiel zu komplexer Division

$z_1 = 1 + 2i$

$z_2 = 1 - i$

$z = \frac{z_1}{z_2} = ?$

$z = \frac{1+2i}{1-i} \quad \Big| \cdot \frac{z_2^*}{z_2^*}$ $\qquad z_2^* z_2 = (1-i)(1+i) = |z_2|^2$

$= \frac{(1+2i)(1+i)}{(1-i)(1+i)} = \frac{1+2ii+2i+i}{1^2+(-1)^2} = \frac{-1+3i}{2}$

$= -\frac{1}{2} + \frac{3}{2}i$

$1+2i+i+2i^2$

Untere Klammer $(1-i)(1+i)$

$= 1 - ii + i - i = 1 - i^2$

$= 1 + 1$

Übung

1) $i(3-2i) = 3i - 2i^2$

$= 3i - 2\cdot(-1) = 3i + 2$

$= 2 + 3i$

Realteil $\qquad \mathrm{Im}(z) = 3$

2) $|3-4i| = \sqrt{3^2+4^2} = \sqrt{25} = \underline{\underline{5}}$

3) $\frac{3+4i}{2-i} = \frac{(3+4i)(2+i)}{(2-i)(2+i)} = \frac{6+8i+3i+4i^2}{2^2+(-1)^2}$

$= \frac{6-4+11i}{5} = \underline{\underline{\frac{2}{5} + \frac{11}{5}i}}$

$= \frac{2+11i}{5}$

Begründung für $e^{i\varphi} = \cos\varphi + i\sin\varphi$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots \frac{x^n}{n!} + \dots$$

Setze $x = i\varphi$

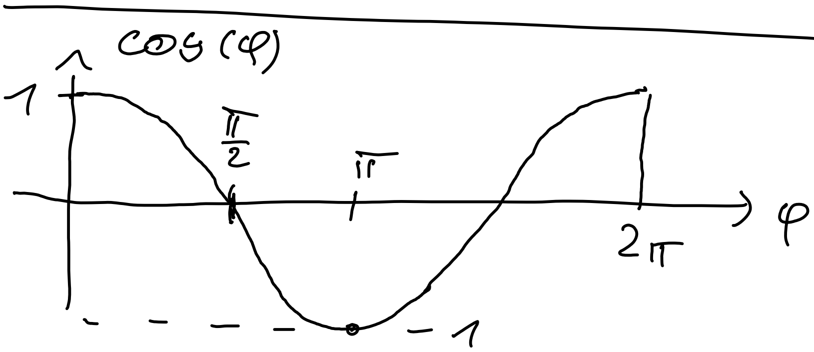
$$e^{i\varphi} = 1 + \frac{i\varphi}{1!} + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^4}{4!} + \dots$$

$$= 1 + i\frac{\varphi}{1!} + \underbrace{i^2}_{=-1}\frac{\varphi^2}{2!} + \underbrace{i^3}_{=-i}\frac{\varphi^3}{3!} + \underbrace{i^4}_{=1}\frac{\varphi^4}{4!} + \dots$$

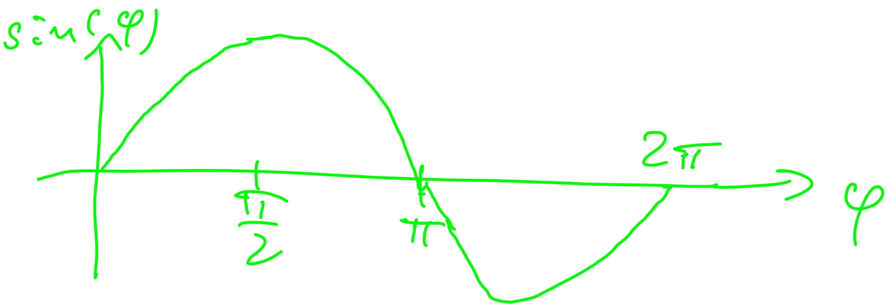
$$= 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + \dots) \quad \text{Taylor f. } \cos\varphi$$

$$+ i\left(\frac{\varphi}{1!} - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} + \dots\right) \quad \text{Taylor f. } \sin\varphi$$

$$\boxed{e^{i\varphi} = \cos\varphi + i\sin\varphi}$$



Radiant	Grad
$\frac{\pi}{2}$	$90°$
π	$180°$
2π	$360°$

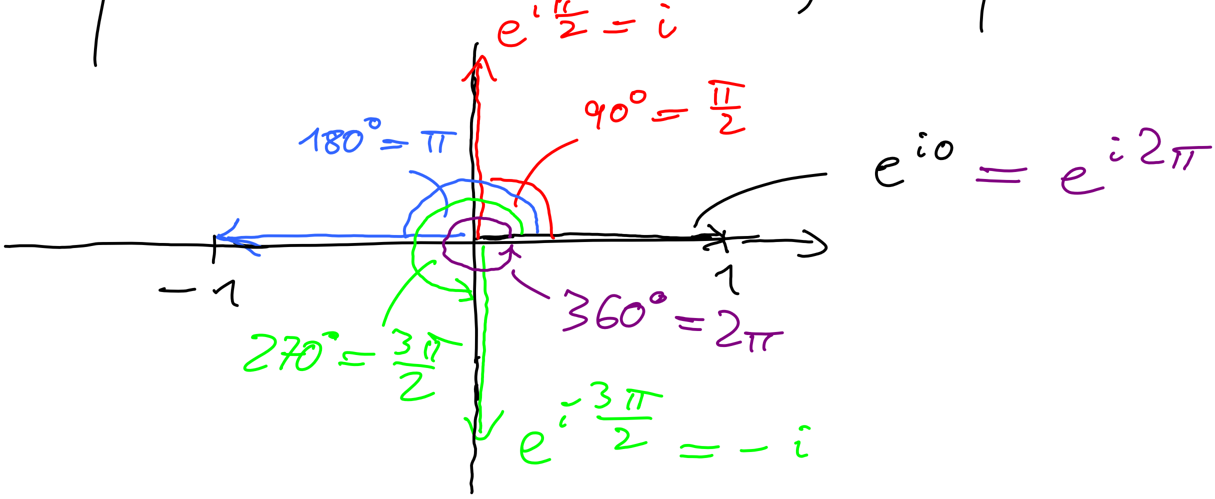


$$\boxed{e^{i\varphi} = \cos\varphi + i\sin\varphi}$$

Tabelle

z	$Re(z) = \cos\varphi$	$Im(z) = \sin(\varphi)$	
e^{i0}	$\cos(0) = 1$	$\sin(0) = 0$	$e^{i0} = 1$
$e^{i\pi}$	$\cos(\pi) = -1$	$\sin(\pi) = 0$	$e^{i\pi} = -1$

z	$Re(z)$	$Im(z)$	
$e^{i2\pi}$	$\cos(2\pi) = 1$	$\sin(2\pi) = 0$	$e^{i2\pi} = 1 = e^{i0}$
$e^{i\frac{\pi}{2}}$	$\cos\left(\frac{\pi}{2}\right) = 0$	$\sin\left(\frac{\pi}{2}\right) = 1$	$e^{i\frac{\pi}{2}} = i$
$e^{i\frac{3\pi}{2}}$	$\cos\left(\frac{3\pi}{2}\right) = 0$	$\sin\left(\frac{3\pi}{2}\right) = -1$	$e^{i\frac{3\pi}{2}} = -i$



$$e^{i0} = e^{i2\pi} = e^{i4\pi} = \ldots = e^{ik2\pi}, \quad k \in \mathbb{Z}$$

$$e^{i\frac{\pi}{2}} = e^{i\left(\frac{\pi}{2} + 2\pi\right)} = e^{i\frac{5\pi}{2}} = \ldots$$