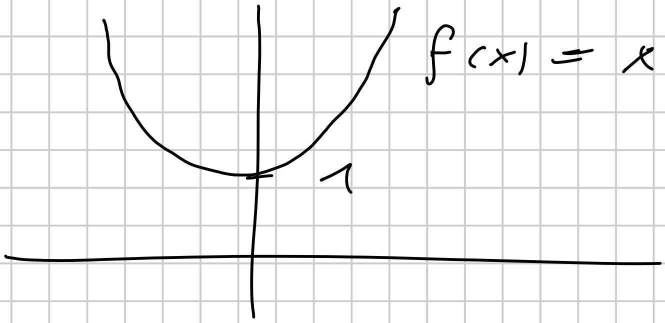
$$\boxed{\text{V MA2 – 20.5.2019}}$$

Komplexe Zahlen



$f(x) = x^2 + 1$

$x^2 + 1 = 0$

keine Lösung in $\mathbb{R}$

$x^2 = -1$ Lösung $x = i \vee x = -i$

$x^2 = (-i)(-i) = i^2 = -1$

$x^2 = -4$ Lösung $x = 2i$

$\vee\ x = -2i$

denn $(-2i)(-2i) = 4i^2 = \underline{\underline{-4}}$

Rechnen mit komplexen Zahlen

$$a_1 + ib_1 \pm (a_2 + ib_2)$$

$$\underbrace{(a_1 \pm a_2)}_{Re(z_1 \pm z_2)} + i \underbrace{(b_1 \pm b_2)}_{Im(z_1 \pm z_2)}$$

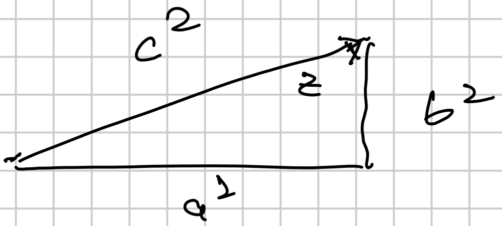
Multiplikation

$$z_1 \cdot z_2 = (a_1 + ib_1) \cdot (a_2 + ib_2)$$

$$= a_1 a_2 + ib_1 a_2 + a_1 i b_2 + \underbrace{ii}_{i^2 = -1} b_1 b^2$$

$$= \underbrace{(a_1 a_2 - b_1 b_2)}_{Re(z_1 \cdot z_2)} + i \underbrace{(a_1 b_2 + a_2 b_1)}_{Im(z_1 z_2)}$$

Betrag komplexe Zahl



$$z = a + ib$$

$$z \cdot z^* = a^2 + b^2$$

$$\hat{=}\ c^2 \quad \text{(Pythagoras)}$$

Division

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{(a_1 + ib_1)\cdot(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}$$

$$= \frac{a_1 a_2 + b_1 b_2 + i(a_2 b_1 - a_1 b_2)}{\underbrace{a_2^2 + b_2^2}_{\text{rein reell}}}$$

Bsp.:

$$\frac{z_1}{z_2} = \frac{1+2i}{1-i} = \frac{(1+2i)(1+i)}{\underbrace{(1-i)}_{z_2}\underbrace{(1+i)}_{z_2^*}} = \frac{1+2i^2+i(2+1)}{1^2+1^2}$$

$$= \frac{-1+i3}{2} = \underbrace{-\frac{1}{2}}_{Re\left(\frac{z_1}{z_2}\right)} + i\underbrace{\frac{3}{2}}_{Im\left(\frac{z_1}{z_2}\right)}$$

Übung

a) $i(3-2i)$ b) $|3-4i|$ c) $\dfrac{3+4i}{2-i}$

a) $z = i(3-2i) = 3i - 2ii = 3i - 2(-1)$

$$= \underbrace{3}_{Im(z)}i + \underbrace{2}_{Re(z)}$$

b) $|z| = |\underbrace{a}_{Re(z)} + i\underbrace{b}_{Im(z)}| = \sqrt{a^2 + b^2}$

$$|3 \underbrace{-4}_{Im} i| = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = \underline{\underline{5}}$$

c) $\frac{3+4i}{2-i} = z$

$$= \frac{(3+4i)(2+i)}{(2-i)(2+i)}$$

$$= \frac{6-4+i(8+3)}{2^2+1^2}$$

$$= \frac{2+11i}{5}$$

$$= \underbrace{\frac{2}{5}}_{Re(z)} + i\,\underbrace{\frac{11}{5}}_{Im(z)}$$

NR
$-4 = 4i \cdot i$

N.R. Analogie

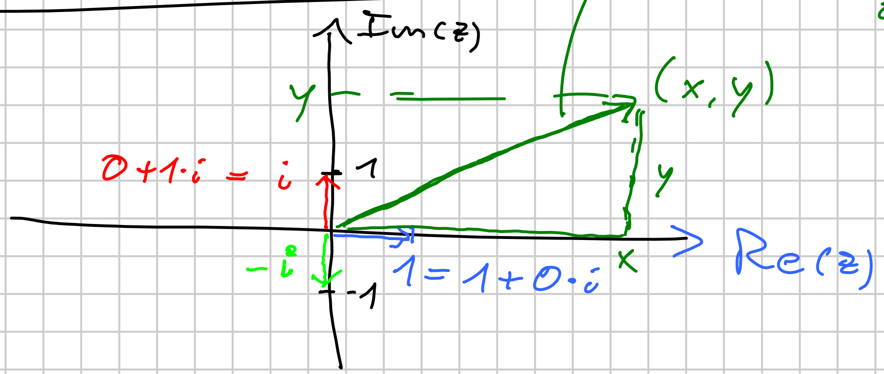
$$\left|\begin{pmatrix}1\\5\end{pmatrix}\right| = \sqrt{1^2+5^2} = \sqrt{26}$$

Gauß-Ebene

Länge Vektor

$$|z| = \sqrt{x^2+y^2}$$

$$z = x+iy$$



Begründung Euler'sche Formel

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots \frac{x^n}{n!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots$$

$x = i\varphi$:

$$e^{i\varphi} = 1 + \frac{i\varphi}{1!} + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \frac{(i\varphi)^4}{4!} + \dots$$

$$= 1 + i\frac{\varphi}{1!} - \frac{\varphi^2}{2!} - i\frac{\varphi^3}{3!} + \frac{\varphi^4}{4!} + \dots$$

$$= \underbrace{\left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \dots\right)}_{\cos\varphi} + i\underbrace{\left(\frac{\varphi}{1!} - \frac{\varphi^3}{3!} + \dots\right)}_{\sin\varphi}$$

$i^3 = i^2 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$$\boxed{e^{i\varphi} = \cos\varphi + i\sin\varphi}$$

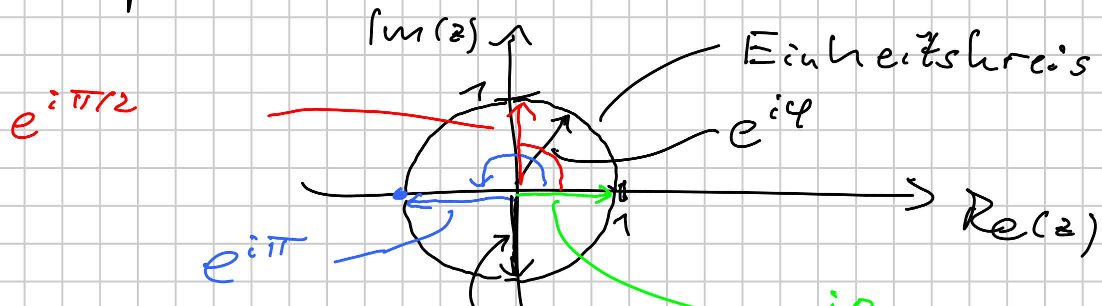
Folgerung: $e^{-i\varphi} = e^{i(-\varphi)} = \cos(-\varphi) + i\sin(-\varphi)$

$$\Rightarrow \boxed{e^{-i\varphi} = \cos\varphi - i\sin\varphi}$$

$$|e^{i\varphi}| = |\cos\varphi + i\sin\varphi| = \sqrt{\cos^2\varphi + \sin^2\varphi} = \sqrt{1} = 1$$

(trigonom. Pythagoras)

$\Rightarrow$ Jede komplexe Zahl $e^{i\varphi}$ hat Länge 1



$e^{i0} = 1 + i0 = e^{i2\pi}$

Übung

z	Re(z)	Im(z)	
e^{i0}	1	0	$e^{i0} = 1$
$e^{i\pi}$	-1	0	$e^{i\pi} = -1$
$e^{i2\pi}$	1	0	$e^{i2\pi} = 1$

N.R.

$\pi \mathrel{\hat{=}} 180°$

$2\pi \mathrel{\hat{=}} 360° = 0°$

$\frac{\pi}{2} = 90°$

$e^{i\pi/2}$	0	1
$e^{-i\pi/2}$	0	-1

$e^{i\pi/2} = i$

$e^{-i\pi/2} = -i$

$-\frac{\pi}{2} = -90^\circ$

$$e^{i\pi/2} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = \dots$$